A Fast Singly Diagonally Implicit Runge–Kutta Method for Solving 1D Unsteady Convection-Diffusion Equations

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In this article, a fast singly diagonally implicit Runge–Kutta method is designed to solve unsteady one-dimensional convection diffusion equations. We use a three point compact finite difference approximation for the spatial discretization and also a three-stage singly diagonally implicit Runge–Kutta (RK) method for the temporal discretization. In particular, a formulation evaluating the boundary values assigned to the internal stages for the RK method is derived so that a phenomenon of the order of the reduction for the convergence does not occur. The proposed scheme not only has fourth-order accuracy in both space and time variables but also is computationally efficient, requiring only a linear matrix solver for a tridiagonal matrix system. It is also shown that the proposed scheme is unconditionally stable and suitable for stiff problems. Several numerical examples are solved by the new scheme and the numerical efficiency and superiority of it are compared with the numerical results obtained by other methods in the literature. Numer Methods Partial Differential Eq 30: 788–812, 2014

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I. INTRODUCTION

In this article, we consider the following unsteady one-dimensional (1D) convection-diffusion equation

$$u_t + \alpha u_x = \beta u_{xx}, \quad (x,t) \in (x_L, x_R) \times (0, T], \tag{1.1}$$

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with the following initial and boundary conditions:

$$u(x, 0) = \phi(x), \quad x \in [x_L, x_R], \tag{1.2}$$

$$u(x_L, t) = g_1(t), \quad u(x_R, t) = g_2(t), \quad t \in (0, T], \tag{1.3}$$

where $u(x, t)$ represents a scalar variable which is convected in the x-direction with constant velocity and is spread with constant diffusivity $\beta > 0, \alpha$ is an arbitrary constant (setting $\alpha = 0$ leads to the usual heat equation) and ϕ is given a sufficiently smooth function. Here, we assume that $g_1(t)$ and $g_2(t)$ are given sufficiently smooth nonconstant functions.

The convection-diffusion equation is one of the most important models in various physical systems. It describes convection and diffusion of quantities such as mass, heat, energy, vorticity, and so forth.(see e.g., [1–5]). Developing stable and effective numerical methods with high accuracy for solving the Eq. (1.1) is very important because an analytical solution of the problem (1.1)–(1.3) can be obtained only with special initial and boundary conditions. These subjects are the main purpose of this work.

In the last decade, many authors have introduced finite difference methods to solve Eq. (1.1). Among these methods, the Crank–Nicolson method (CNM) is known as the most widely used and popular one, which is based on the classical trapezoidal formula for integration in time and the central difference formula for spatial discretization. It is well-known that the CNM is unconditionally stable, but has only a second order of accurate in both time and space. Hence, it is necessary to develop a computationally efficient method with high accuracy.

The application of compact finite difference methods for the discretization of the spatial variables of the problem (1.1) has attracted recent interest. It achieves high spatial accuracy with a small stencil (see e.g., [6–12]). As a high-order method, Ding and Zhang [7] developed a different scheme using semidiscrete and Padé approximation methods for the temporal variable and a polynomial compact difference discretization for the spatial variable outlined in [9]. It has a fourth-order convergence rate in both time and space and is unconditionally stable. However, the fourth-order compact formula [7] is a lower resolution scheme. Therefore, it is not suitable to solve convection-dominated problems (see e.g., [12, 13]). Tian and Yu [14] introduced a high-order exponential (HOE) method for solving (1.1) which uses the (2,2) Padé approximation to the exponential function for the temporal variable and the fourth-order compact finite difference formula [12] for the spatial variable. The HOE scheme turned out to be not only unconditionally stable but also have a fourth-order convergence rate in both time and space. Most recently, by using a rational approximation for the exponential function outlined in [15] instead of (2,2) Padé approximation, the authors [16] improved efficiently the HOE scheme requiring only a liner matrix solver for a tridiagonal matrix system. However, two methods aforementioned are not suitable to solve (1.1) with nonconstant boundary conditions. In the case of the nonconstant boundary conditions, Mohebbi and Dehghan [17] introduced a high-order accurate method based on a fourth-order compact finite difference approximation for the spatial variable and a cubic C^1-spline collocation method (CSCM) for the temporal variable. It is an A-stable method and has fourth-order accuracy in both spatial and temporal variables. Both (2,2) Padé approximation [12] and the cubic CSCM [17] require a costly matrix solver for either complex tridiagonal systems or quadratic polynomial systems consisting of tridiagonal matrices.

The primary goal of this article is to construct a fast singly diagonally implicit Runge–Kutta (FSDIRK) method which has a high-order accuracy and does not require such a costly solver for complex tridiagonal systems as in the (2,2) Padé approximation [12] and the cubic CSCM [17]. These objects can be obtained by applying a three point compact finite difference scheme for the space variable and a three-stage singly diagonally implicit Runge–Kutta (SDIRK) method for the temporal variable. In particular, we develop a technique calculating the unknown boundary conditions assigned to the internal stages for the SDIRK method which does not lead such phenomenon of the order of the reduction for the convergence occurring in most methods of lines for the case of stiff boundary conditions. It is proved that the scheme has a fourth-order convergence in both spatial and temporal variables and is unconditionally stable. Furthermore, the whole algorithm of the FSDIRK method requires only a fast linear solver for tridiagonal matrix systems.

This article is organized as follows. In Section II, we review and clarify the three point fourth-order compact finite difference scheme for the spatial derivatives developed by [14]. In Section III, we review the three-stage SDIRK method and derive a formula that can be used to evaluate the unknown boundary conditions assigned to the internal stages of the RK method. In Section IV, a high-order FSDIRK scheme for solving (1.1) is derived using the results of Section II and Section III. The concrete analysis of convergence and stability is described in Section V and VI, respectively. Several test problems are solved in Section VII to give numerical evidence for the theoretical analysis. Also, we show the efficiency of the proposed method by comparing it with other existing methods. Finally, we close the article by summarizing the results in Section VIII.

II. A FOURTH-ORDER COMPACT FINITE DIFFERENCE SCHEME

Several different techniques for deriving a fourth-order compact finite difference scheme can be found in many literature. For example, see [5, 8, 12, 14, 17] and references therein. In this section, we review and clarify the three point fourth-order compact finite difference scheme for the spatial derivatives of (1.1) developed by [14], recently. For a positive integer n, let $h = \frac{x_R - x_L}{n}$ denote the step size for spatial variable x and k for the step size of temporal variable. Then, we define

$$x_i = x_L + ih, \quad i = 0, 1, 2, \cdots, n,$$
$$t_j = jk, \quad j = 0, 1, 2, \cdots.$$

We consider the following two-point boundary value problem in each subdomain $[x_{i-1}, x_{i+1}]$ $(i = 1, 2, \cdots, n-1)$:

$$\begin{cases} -\beta u_{xx} + \alpha u_x = f, & x \in (x_{i-1}, x_{i+1}), \\ u(x_{i-1}) = u_{i-1}, & u(x_{i+1}) = u_{i+1}, \end{cases} \tag{2.1}$$

where f is a sufficiently smooth function of x. Then, by the Green function method (see e.g., [18, 19]), the solution of (2.1) can be expressed by

$$u(x) = \varphi_1(x)u_{i-1} + \varphi_2(x)u_{i+1} + \int_{x_{i-1}}^{x_{i+1}} G(x, \eta) f(\eta) d\eta, \tag{2.2}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are two solutions of the equation $-\beta\varphi_{xx} + \alpha\varphi_x = 0$ that satisfy the boundary conditions

$$\varphi(x_{i-1}) = 1, \quad \varphi(x_{i+1}) = 0$$

and

$$\varphi(x_{i-1}) = 0, \quad \varphi(x_{i+1}) = 1,$$

respectively, and $G(x,\eta)$ is the Green function of the problem

$$\begin{cases} -\beta u_{xx} + \alpha u_x = 0, \quad x \in (x_{i-1}, x_{i+1}), \\ u(x_{i-1}) = 0, \quad u(x_{i+1}) = 0. \end{cases}$$

By some manipulation, one can find the explicit forms of $\varphi_j(x)(j = 1,2)$ and $G(x,\eta)$ as follows (see e.g., [20]):

$$\varphi_1(x) = \frac{1 - e^{\frac{\alpha}{\beta}(x - x_{i+1})}}{1 - e^{-2\frac{\alpha}{\beta}h}}, \quad \varphi_2(x) = \frac{e^{\frac{\alpha}{\beta}(x - x_{i-1})} - 1}{e^{2\frac{\alpha}{\beta}h} - 1}, \quad G(x,\eta) = \frac{1}{W(\eta)} \begin{cases} \varphi_1(x)\varphi_2(\eta), & \eta < x, \\ \varphi_1(\eta)\varphi_2(x), & x \le \eta, \end{cases} \tag{2.3}$$

where the Wronskian $W(\eta)$ is given by

$$W(\eta) = \frac{\alpha e^{\frac{\alpha}{\beta}(\eta - x_i)}}{e^{\frac{\alpha}{\beta}h} - e^{-\frac{\alpha}{\beta}h}}. \tag{2.4}$$

Evaluating $\varphi_j(x)(j = 1,2)$ at the point x_i, we get $\varphi_1(x_i) = (1 + e^{-\frac{\alpha}{\beta}h})^{-1}$ and $\varphi_2(x_i) = (1 + e^{\frac{\alpha}{\beta}h})^{-1}$. Hence, by (2.3), (2.4), and a change of variables, the Eq. (2.2) gives

$$u(x_i) = \frac{u_{i-1}}{1 + e^{-\frac{\alpha}{\beta}h}} + \frac{u_{i+1}}{1 + e^{\frac{\alpha}{\beta}h}} + \frac{1}{\alpha}\int_0^h \left(\frac{1 - e^{-\frac{\alpha}{\beta}t}}{1 + e^{-\frac{\alpha}{\beta}h}} f(t + x_{i-1}) + \frac{e^{\frac{\alpha}{\beta}t} - 1}{1 + e^{\frac{\alpha}{\beta}h}} f(x_{i+1} - t) \right) dt. \tag{2.5}$$

Let $\delta_x f_i$ and $\delta_x^2 f_i$ be the standard second-order central differential operators defined by

$$\delta_x f_i := \frac{f(x_{i+1}) - f(x_{i-1})}{2h}, \quad \delta_x^2 f_i := \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}, \quad i = 1,2,\cdots,n-1. \tag{2.6}$$

Then, by the Taylor's expansion for f about $x = x_i$, one can easily check that

$$\delta_x f_i = f'(x_i) + \frac{h^2}{6} f^{(3)}(x_i) + O(h^4), \quad \delta_x^2 f_i = f''(x_i) + O(h^2). \tag{2.7}$$

These asymptotic behaviors in (2.7) leads to the following lemma.

Lemma 2.1. *The central differential operators $\delta_x f_i$ and $\delta_x^2 f_i$ satisfy the following relation*

$$f(\eta) = f(x_i) + (\eta - x_i)\delta_x f_i + \frac{(\eta - x_i)^2}{2}\delta_x^2 f_i + \frac{f^{(3)}(x_i)}{6}(\eta - x_{i-1})(\eta - x_i)(\eta - x_{i+1}) + O(h^4), \quad \eta \in (x_{i-1}, x_{i+1}).$$

Proof. Since the Taylor's expansion of $f(\eta)$ about x_i is given by

$$f(\eta) = f(x_i) + f'(x_i)(\eta - x_i) + f''(x_i)\frac{(\eta - x_i)^2}{2} + f^{(3)}(x_i)\frac{(\eta - x_i)^3}{6} + O(h^4), \quad \eta \in (x_{i-1}, x_{i+1}), \tag{2.8}$$

combining (2.8) with (2.7) leads to

$$f(\eta) = f(x_i) + \left(\delta_x f_i - \frac{h^2}{6} f^{(3)}(x_i)\right)(\eta - x_i) + \delta_x^2 f_i \frac{(\eta - x_i)^2}{2} + \frac{f^{(3)}(x_i)}{6}(\eta - x_i)^3 + O(h^4)$$

$$= f(x_i) + \delta_x f_i(\eta - x_i) + \delta_x^2 f_i \frac{(\eta - x_i)^2}{2} + \frac{f^{(3)}(x_i)}{6}(\eta - x_i)((\eta - x_i)^2 - h^2) + O(h^4)$$

which yields the required relation by the fact $(\eta - x_i)^2 - h^2 = (\eta - x_{i-1})(\eta - x_{i+1})$. $\square$

Note that

$$\frac{1 - e^{-\frac{\alpha}{\beta}t}}{1 + e^{-\frac{\alpha}{\beta}h}} + \frac{e^{\frac{\alpha}{\beta}t} - 1}{1 + e^{\frac{\alpha}{\beta}h}} = 2\,\mathrm{sech}\,\frac{\alpha h}{2\beta}\cosh\frac{\alpha(h - t)}{2\beta}\sinh\frac{\alpha t}{2\beta}$$

$$\frac{1 - e^{-\frac{\alpha}{\beta}t}}{1 + e^{-\frac{\alpha}{\beta}h}} - \frac{e^{\frac{\alpha}{\beta}t} - 1}{1 + e^{\frac{\alpha}{\beta}h}} = 1 - \cosh\frac{\alpha(h - 2t)}{2\beta}\mathrm{sech}\,\frac{\alpha h}{2\beta},$$

Also, by a direct integration and the Taylor's expansion, one may see that

$$\int_0^h \frac{1 - e^{\pm\frac{\alpha}{\beta}t}}{1 + e^{\pm\frac{\alpha}{\beta}h}} dt = \frac{h}{1 + e^{\pm\frac{\alpha}{\beta}h}} \pm \frac{\beta}{\alpha}\frac{1 - e^{\pm\frac{\alpha}{\beta}h}}{1 + e^{\pm\frac{\alpha}{\beta}h}} = \pm\frac{\alpha h^2}{4\beta} + O(h^3).$$

Hence, combining (2.5) with Lemma 2.1 leads to

$$\begin{aligned}
u(x_i) &= \frac{u_{i-1}}{1 + e^{-\frac{\alpha}{\beta}h}} + \frac{u_{i+1}}{1 + e^{\frac{\alpha}{\beta}h}} \\
&\quad + \frac{1}{\alpha}\int_0^h \Bigg((2f_i + (t - h)^2\delta_x^2 f_i)\mathrm{sech}\,\frac{\alpha h}{2\beta}\cosh\frac{\alpha(h - t)}{2\beta}\sinh\frac{\alpha t}{2\beta} \\
&\quad + \left((t - h)\delta_i f_i + \frac{f^{(3)}(x_i)}{6} t(t - h)(t - 2h)\right) \\
&\quad \times \left(1 - \cosh\frac{\alpha(h - 2t)}{2\beta}\mathrm{sech}\,\frac{\alpha h}{2\beta}\right)\Bigg) dt + O(h^6) \\
&= \frac{u_{i-1}}{1 + e^{-\frac{\alpha}{\beta}h}} + \frac{u_{i+1}}{1 + e^{\frac{\alpha}{\beta}h}} + \frac{1}{\alpha}\Bigg(h\tanh\left(\frac{h\alpha}{2\beta}\right) f_i + \delta_x f_i\left(\frac{h\beta}{\alpha}\tanh\frac{h\alpha}{2\beta} - \frac{h^2}{2}\right) \\
&\quad + \delta_x^2 f_i \frac{\beta^2 h\left(-3h\frac{\alpha}{\beta} + \left(6 + h^2\left(\frac{\alpha}{\beta}\right)^2\right)\tanh\frac{h\alpha}{2\beta}\right)}{6\alpha^2} \\
&\quad + \frac{f^{(3)}(x_i)}{6}\left(\frac{h^4}{4} - \frac{3(\beta h)^2}{\alpha^2} + \frac{6h\beta^3\tanh\frac{h\alpha}{2\beta}}{\alpha^3}\right)\Bigg) + O(h^6).
\end{aligned} \tag{2.9}$$

Since the Taylor's expansion of $\tanh x$ is given by

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \frac{62x^9}{2835} + O(x^{11}),$$

by dividing (2.9) by $\frac{h}{\alpha}\tanh\frac{h\alpha}{2\beta}$ and rearranging the results, we get

$$-a\delta_x^2 u_i + \alpha\delta_x u_i = f_i + a_1\delta_x f_i + a_2\delta_x^2 f_i + O(h^4). \tag{2.10}$$

where

$$a := \frac{\alpha h}{2}\coth\frac{h\alpha}{2\beta}, \quad a_1 := \frac{\beta - a}{\alpha}, \quad a_2 := \frac{\beta(\beta - a)}{\alpha^2} + \frac{h^2}{6}. \tag{2.11}$$

When $\alpha = 0$, the first equation of (2.1) reduces to

$$-\beta u_{xx} = f. \tag{2.12}$$

The fourth-order Pad é approximation for the second-order derivative gives

$$(u_{xx})_i = \frac{\delta_x^2}{1 + \frac{h^2}{12}\delta_x^2} u_i + O(h^4). \tag{2.13}$$

Substituting (2.13) into (2.12) leads to

$$-\beta\delta_x^2 u_i = \left(1 + \frac{h^2}{12}\delta_x^2\right) f_i + O(h^4). \tag{2.14}$$

Combining (2.10) with (2.14) and omitting the truncation error, a generalized three-point fourth-order compact finite difference formulation for the problem (2.1) can be obtained by

$$(-a\delta_x^2 + \alpha\delta_x)u_i = (1 + a_1\delta_x + a_2\delta_x^2) f_i, \tag{2.15}$$

where

$$a := \begin{cases} \frac{\alpha h}{2}\coth\frac{h\alpha}{2\beta}, & \alpha \neq 0, \\ \beta, & \alpha = 0, \end{cases} \qquad a_1 := \begin{cases} \frac{\beta - a}{\alpha}, & \alpha \neq 0, \\ 0, & \alpha = 0, \end{cases} \qquad a_2 := \begin{cases} \frac{\beta(\beta - a)}{\alpha^2} + \frac{h^2}{6}, & \alpha \neq 0, \\ \frac{h^2}{12}, & \alpha = 0. \end{cases} \tag{2.16}$$

■

III. IMPLICIT RUNGE–KUTTA METHOD

It is well-known that a direct usage of the SDIRK method for solving an initial boundary value problem leads to unknown boundary conditions assigned to the internal stages. In the method of lines, these conditions yield a phenomenon of the reduction of convergence order for the case of a stiff boundary condition. The aim of this section is to develop a formulation that can be used to determine the unknown boundary conditions for the internal stages by clarifying some properties of SDIRK. We consider an autonomous equation

$$y'(t) = f(y(t)) \tag{3.1}$$

and for its discretization, we consider the three stage RK method with Butcher array

$$\begin{array}{c|c} \mathbf{c} & \mathcal{A} \\ \hline & \mathbf{b}^T \end{array}, \tag{3.2}$$

where

$$\mathbf{c} = [c_1, c_2, c_3]^T := \left[r, \frac{1}{2}, 1 - r \right]^T,$$

$$\mathbf{b} = [b_1, b_2, b_3]^T := \left[\frac{1}{6(1-2r)^2}, 1 - \frac{1}{3(1-2r)^2}, \frac{1}{6(1-2r)^2} \right]^T,$$

$$\mathcal{A} := (a_{ij}) := \begin{bmatrix} r & 0 & 0 \\ \frac{1}{2} - r & r & 0 \\ 2r & 1 - 4r & r \end{bmatrix} \tag{3.3}$$

with $r = \frac{1}{2} + \frac{\sqrt{3}}{3}\cos(\frac{\pi}{18})$. Then, we recall the three-stage fourth-order SDIRK for solving (3.1) stated as

$$K_i = f\left(y_m + k \sum_{j=1}^{i} a_{ij} K_j \right), \quad i = 1, 2, 3,$$

$$y_{m+1} = y_m + k \sum_{i=1}^{3} b_i K_i, \tag{3.4}$$

where y_m is the approximation of $y(t)$ at time t_m and k is a fixed time step size. By the Taylor's expansion of $f(y_m + k \sum_{j=1}^{i} a_{ij} K_j)$ about y_m, one may rewrite K_i of (3.4) by

$$K_i = f\left(y_m + k \sum_{j=1}^{i} a_{ij} K_j \right) = f + kX_i f_y + \frac{k^2}{2} X_i^2 f_{yy} + \frac{k^3}{6} X_i^3 f_{yyy} + O(k^4), \quad i = 1, 2, 3, \tag{3.5}$$

where all the functions in the right hand side are evaluated at the point y_m and

$$X_i = \sum_{j=1}^{i} a_{ij} K_j. \tag{3.6}$$

From the above two relations (3.5) and (3.6), we have the following lemma.

Lemma 3.1. *The quantity X_i defined by (3.6) satisfy the following relation*

$$\mathbf{X} = f\mathbf{c} + kff_y\mathcal{A}\mathbf{c} + k^2\left(ff_y^2\mathcal{A}^2\mathbf{c} + \frac{1}{2} f^2 f_{yy}\mathcal{A}\mathbf{c}^2 \right) + O(k^3), \tag{3.7}$$

where $\mathbf{X} = [X_1, X_2, X_3]^T$ *and* $\mathbf{c}^2 = [c_1^2, c_2^2, c_3^3]^T$.

Proof. By combining (3.6) with (3.5) and using the identity $\sum_{j=1}^{i} a_{ij} = c_i (i = 1, 2, 3)$, one may have

$$\mathbf{X} = f\mathbf{c} + kf_y\mathcal{A}\mathbf{X} + \frac{k^2}{2} f_{yy}\mathcal{A}\mathbf{X}^2 + O(k^3). \tag{3.8}$$

Plugging the whole expression (3.8) back in for $\mathbf{X}$ and removing terms of truncation order $O(k^3)$, one get

$$\mathbf{X} = f\mathbf{c} + kf_y\mathcal{A}(f\mathbf{c} + kf_y\mathcal{A}\mathbf{X}) + \frac{k^2}{2}f_{yy}f^2\mathcal{A}\mathbf{c}^2 + O(k^3)$$
$$= f\mathbf{c} + kff_y\mathcal{A}\mathbf{c} + k^2f_y^2\mathcal{A}^2(f\mathbf{c}) + \frac{k^2}{2}f_{yy}f^2\mathcal{A}\mathbf{c}^2 + O(k^3),$$

which gives the required expression (3.7).□

From (3.3), we can easily check that

$$(\mathcal{A}\mathbf{1})_i = c_i, \quad i = 1, 2, 3,$$

where $\mathbf{1}$ denotes the vector defined by $\mathbf{1} := [1, 1, 1]^T$. Thus, by combining (3.7) with (3.5), we have the following Corollary. ■

Corollary 3.2. *The internal stages K_i defined by (3.4) satisfy the following expression*

$$K_i = f + k(\mathcal{A}\mathbf{1})_i f_y f + k^2\left((\mathcal{A}\mathbf{c})_i f_y^2 f + \frac{c_i^2}{2}f_{yy}f^2\right)$$
$$+k^3\left((\mathcal{A}^2\mathbf{c})_i f_y^3 f + \left(\frac{1}{2}(\mathcal{A}\mathbf{c}^2)_i + c_i(\mathcal{A}\mathbf{c})_i\right)f_y f_{yy} f^2 + \frac{c_i^3}{6}f_{yyy}f^3\right) + O(k^4), \quad i = 1, 2, 3, \tag{3.9}$$

where $(\mathbf{a})_i$ denotes the i-th component of a vector $\mathbf{a}$.

The Corollary 3.2 gives the following lemma.

Lemma 3.3. *Assume that f is a linear function defined by $f(y) = \lambda y$ with a constant λ and $\{y_m\}$ is the approximation solution of (3.1) with a fourth-order convergence. Then, there are a relation between the solution of (3.1) and the internal stages K_i such that*

$$K_i = y'(t_m) + k(\mathcal{A}\mathbf{1})_i y''(t_m) + k^2(\mathcal{A}^2\mathbf{1})_i y^{(3)}(t_m) + k^3(\mathcal{A}^3\mathbf{1})_i y^{(4)}(t_m) + O(k^4), \quad i = 1, 2, 3. \tag{3.10}$$

Proof. Substituting (3.9) into the first equation of (3.4) and using the linearity of f, we get

$$K_i = f(y_m) + k\sum_{j=1}^{i} a_{ij}\left(f + k(\mathcal{A}\mathbf{1})_j f_y f + k^2\left((\mathcal{A}\mathbf{c})_j f_y^2 f + \frac{c_j^2}{2}f_{yy}f^2\right)\right) + O(k^4)$$
$$= f(y_m) + kf(f)\sum_{j=1}^{i} a_{ij} + k^2 f(f_y f)\sum_{j=1}^{i} a_{ij}(\mathcal{A}\mathbf{1})_j$$
$$+ k^3\left(f(f_y^2 f)\sum_{j=1}^{i} a_{ij}(\mathcal{A}\mathbf{c})_j + f(f_{yy}f^2)\sum_{j=1}^{i} a_{ij}\frac{c_j^2}{2}\right) + O(k^4). \tag{3.11}$$

Since the local truncation error between y_m and $y(t_m)$ is $O(k^5)$ and

$$\sum_{j=1}^{i} a_{ij} = (\mathcal{A}\mathbf{1})_i, \quad \sum_{j=1}^{i} a_{ij}(\mathcal{A}\mathbf{1})_j = (\mathcal{A}^2\mathbf{1})_i, \quad \sum_{j=1}^{i} a_{ij}(\mathcal{A}\mathbf{c})_j = (\mathcal{A}^2\mathbf{c})_i,$$

the equation of (3.11) can be simplified by

$$\begin{aligned} K_i &= f(y_m) + k(\mathcal{A}\mathbf{1})_i f(f) + k^2(\mathcal{A}^2\mathbf{1})_i f(f_y f) \\ &\quad + k^3\left((\mathcal{A}^2\mathbf{c})_i f(f_y^2 f) + \frac{1}{2}(\mathcal{A}\mathbf{c}^2)_i f(f_{yy} f^2)\right) + O(k^4) \\ &= y'(t_m) + ky''(t_m)(\mathcal{A}\mathbf{1})_i + k^2 y^{(3)}(t_m)(\mathcal{A}^2\mathbf{1})_i + O(k^3). \end{aligned} \tag{3.12}$$

Now, by substituting (3.12) into the first equation of (3.4) and using the definition of f, we get

$$\begin{aligned} K_i &= f\left(y_m + k\sum_{j=1}^{i} a_{ij}\left(y'(t_m) + ky''(t_m)(\mathcal{A}\mathbf{1})_j + k^2 y^{(3)}(t_m)(\mathcal{A}^2\mathbf{1})_j\right)\right) + O(k^4) \\ &= y'(t_m) + kf(y'(t_m))\sum_{j=1}^{i} a_{ij} + k^2 f(y''(t_m))\sum_{j=1}^{i} a_{ij}(\mathcal{A}\mathbf{1})_j \\ &\quad + k^3 f(y^{(3)}(t_m))\sum_{j=1}^{i} a_{ij}(\mathcal{A}^2\mathbf{1})_j + O(k^4) \\ &= y'(t_m) + k(\mathcal{A}\mathbf{1})_i y''(t_m) + k^2(\mathcal{A}^2\mathbf{1})_i y^{(3)}(t_m) + k^3(\mathcal{A}^3\mathbf{1})_i y^{(4)}(t_m) + O(k^4), \end{aligned}$$

which completes the proof. ■

IV. THE PROPOSED METHOD

In this section, we will derive a new fast method for solving (1.1) based on the methods described in the previous two sections. We start to discretize the Eq. (1.1) in space to obtain a system of ordinary differential equations with unknown functions at each spatial grid point. Replacing f in (2.15) by $-u_t$ leads to the following relation:

$$\begin{aligned} &\left(\frac{a}{h^2} + \frac{\alpha}{2h}\right)u_{i-1}(t) - \frac{2a}{h^2}u_i(t) + \left(\frac{a}{h^2} - \frac{\alpha}{2h}\right)u_{i+1}(t) \\ &\qquad = \left(\frac{a_2}{h^2} - \frac{a_1}{2h}\right)u'_{i-1}(t) + \left(1 - \frac{2a_2}{h^2}\right)u'_i(t) + \left(\frac{a_2}{h^2} + \frac{a_1}{2h}\right)u'_{i+1}(t), \end{aligned} \tag{4.1}$$

where

$$u_i(t) = u(x_i, t), \quad u'_i(t) = u_t(x_i, t).$$

For simplicity of discussions, let us define two operators δ_A and δ_B as follows:

$$\delta_A u_i'(t) = \left(\frac{a_2}{h^2} - \frac{a_1}{2h}\right) u_{i-1}'(t) + \left(1 - \frac{2a_2}{h^2}\right) u_i'(t) + \left(\frac{a_2}{h^2} + \frac{a_1}{2h}\right) u_{i+1}'(t),$$

$$\delta_B u_i(t) = \left(\frac{a}{h^2} + \frac{\alpha}{2h}\right) u_{i-1}(t) - \frac{2a}{h^2} u_i(t) + \left(\frac{a}{h^2} - \frac{\alpha}{2h}\right) u_{i+1}(t). \tag{4.2}$$

Then, the Eq. (4.1) can be simplified as

$$\delta_A u_i'(t) = \delta_B u_i(t), \quad i = 1, 2, \cdots, n-1. \tag{4.3}$$

For a discretization for the temporal variable, we apply the three-stage fourth-order SDIRK described by (3.4) to (4.3). Then, one may have

$$\delta_A K_{l,i} = \delta_B \left(u_i^m + k \sum_{j=1}^{l} a_{lj} K_{j,i} \right), \quad l = 1, 2, 3, \quad i = 1, 2, \cdots, n-1,$$

$$u_i^{m+1} = u_i^m + k \sum_{j=1}^{3} b_j K_{j,i}, \quad i = 0, 1, \cdots, n, \tag{4.4}$$

where u_i^m denotes an approximation of $u(x_i, t_m)$. Note that the boundary values assigned to the internal stages $K_{j,i} (j = 1, 2, 3, i = 0, n)$ in (4.4) are not specified, which leads to an order of reduction usually and nonunique solution. That is, there needs to specify these boundaries so that the classical order of the RK method is retrieved. Based on Lemma 3.3, we use the following boundary conditions for internal stage:

$$[K_{1,0}, K_{2,0}, K_{3,0}]^T := \sum_{j=1}^{4} k^{j-1} g_1^{(j)}(t_m) \mathcal{A}^{j-1} \mathbf{1} \tag{4.5}$$

and

$$[K_{1,n}, K_{2,n}, K_{3,n}]^T = \sum_{j=1}^{4} k^{j-1} g_2^{(j)}(t_m) \mathcal{A}^{j-1} \mathbf{1}, \tag{4.6}$$

where $g_j^{(k)}(t)$ $(j = 1, 2)$ denote the k-th derivative of $g_j(t)$ with respect to t. Further, to avoid the difficulty of the derivatives and keep the same accuracy of convergence order, we will use the Lagrange interpolation polynomials with six CGL points in $[t_m, t_{m+1}]$ for the functions $g_1(t)$ and $g_2(t)$ instead of the derivatives of g_j $(j = 1, 2)$ in the formula (4.5) and (4.6).

For a convenience, we give a detailed implementation of (4.4) and its solvability in the following. Let $\mathbf{A}$ and $\mathbf{B}$ be $(n-1) \times (n-1)$ tridiagonal matrices defined by

$$\mathbf{A} := \operatorname{tri}\left[\frac{a_2}{h^2} - \frac{a_1}{2h}, 1 - \frac{2a_2}{h^2}, \frac{a_2}{h^2} + \frac{a_1}{2h}\right]_{n-1}, \quad \mathbf{B} := \operatorname{tri}\left[\frac{a}{h^2} + \frac{\alpha}{2h}, -\frac{2a}{h^2}, \frac{a}{h^2} - \frac{\alpha}{2h}\right]_{n-1}, \tag{4.7}$$

where $\operatorname{tri}[c_1, c_2, c_3]_{n-1}$ denotes the $(n-1) \times (n-1)$ tridiagonal matrix whose each row contains the values c_1, c_2, and c_3 on its subdiagonal, diagonal, and superdiagonal, respectively. Then from (4.2), (4.5), and (4.6), the system of (4.4) can be rewritten by

$$(\mathbf{A} - kr\mathbf{B})\tilde{\mathbf{K}}_i = \mathbf{B}\left(\mathbf{U}^m + k \sum_{j=1}^{i-1} a_{i,j} \tilde{\mathbf{K}}_j\right) + \mathbf{R}_i, \quad i = 1, 2, 3, \tag{4.8}$$

where

$$\tilde{\mathbf{K}}_i := [K_{i,1}, K_{i,2}, \cdots, K_{i,n-1}]^T, \quad \mathbf{U}^m := [u_1^m, u_2^m, \cdots, u_{n-1}^m]^T, \quad \mathbf{R}_i := [R_{i,1}, 0, \cdots, 0, R_{i,2}]^T,$$

$$R_{i,1} := \left(\frac{a}{h^2} + \frac{\alpha}{2h}\right)\left(g_1(t_m) + k\sum_{j=1}^{i} a_{i,j} K_{j,0}\right) - \left(\frac{a_2}{h^2} - \frac{a_1}{2h}\right) K_{i,0},$$

$$R_{i,2} := \left(\frac{a}{h^2} + \frac{\alpha}{2h}\right)\left(g_2(t_m) + k\sum_{j=1}^{i} a_{1,1} K_{i,n}\right) - \left(\frac{a_2}{h^2} - \frac{a_1}{2h}\right) K_{i,n}.$$

The solvability of the systems of (4.8) follows from the next lemma.

Lemma 4.1. *Let* $r = \frac{1}{2} + \frac{\sqrt{3}}{3}\cos(\frac{\pi}{18})$ *. Then, for the tridiagonal matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *defined by (4.7), the tridiagonal matrix* $(\mathbf{A} - kr\mathbf{B})$ *is nonsingular.*

Proof. From (2.16) and (4.7), setting $\alpha = 0$ leads to

$$\mathbf{A} - kr\mathbf{B} = \text{tri}\left[\frac{1}{12} - \frac{\beta kr}{h^2}, \frac{5}{6} + \frac{2\beta kr}{h^2}, \frac{1}{12} - \frac{\beta kr}{h^2}\right] \tag{4.9}$$

whose eigenvalues can be calculated as follows (see e.g., [21]):

$$\lambda_j = \frac{5}{6} + \frac{2\beta kr}{h^2} + 2\left|\frac{1}{12} - \frac{\beta kr}{h^2}\right|\cos\frac{\pi j}{n}, \quad j = 1, \cdots, n-1.$$

One may easily check that each eigenvalue λ_j is larger than $\frac{2}{3}$ at least and therefore the matrix $\mathbf{A} - kr\mathbf{B}$ given by (4.9) is nonsingular. Now, we consider the case $\alpha \neq 0$. In the case, due to the inequality $\xi\coth\xi \geq 1, \xi \in (-\infty, \infty)$(see e.g., [5]), the quantities a and a_2 defined by (2.16) can be estimated by

$$a = \frac{\alpha h}{2}\coth\frac{h\gamma}{2} = \beta\left(\frac{\alpha h}{2\beta}\coth\frac{\alpha h}{2\beta}\right) \geq \beta, \quad a_2 = \frac{\beta(\beta - a)}{\alpha^2} + \frac{h^2}{6} \leq \frac{h^2}{6}. \tag{4.10}$$

From (2.16) and (4.7), the eigenvalues of the tridiagonal matrix $\mathbf{A} - kr\mathbf{B}$ are given by [21]

$$\lambda_j = 1 - \frac{2a_2}{h^2} + \frac{2akr}{h^2} + 2\sqrt{\Delta}\cos\frac{\pi j}{n}, \quad \Delta = \left(\frac{a_2}{h^2} - \frac{akr}{h^2}\right)^2 - \left(\frac{a_1}{2h} + \frac{\alpha kr}{2h}\right)^2, \quad j = 1, \cdots, n. \tag{4.11}$$

For the case $\Delta < 0$, each eigenvalue λ_j can be written as $\lambda_j = x_j + iy_j$ and hence from (4.10), one can estimated x_j as follows:

$$x_j = 1 - \frac{2a_2}{h^2} + \frac{2akr}{h^2} \geq \frac{2}{3} + \frac{2\beta kr}{h^2} > \frac{2}{3}.$$

For the case $\Delta \geq 0$, if one use the fact

$$1 - \left(\frac{a_1}{2h} + \frac{\alpha kr}{2h}\right)^2\left(\frac{a_2}{h^2} - \frac{akr}{h^2}\right)^{-2} < 1,$$

one can easily estimate each eigenvalue in (4.11) as follows:

$$\lambda_j > 1 - \left(\frac{2a_2}{h^2} - \frac{2akr}{h^2}\right) - 2\left|\frac{a_2}{h^2} - \frac{akr}{h^2}\right| > \frac{1}{3}, \quad j = 1, \cdots, n-1.$$

Hence, one can find that all eigenvalues of $\mathbf{A} - kr\mathbf{B}$ are not zero in any cases and hence $\mathbf{A} - kr\mathbf{B}$ is nonsingular. ■

Based on Lemma 4.1, after solving the systems (4.8), one can calculate approximations u_i^{m+1} for the solution $u(t_i, x_{m+1})$ uniquely from the second equations of (4.4). Note that since $\mathbf{A} - kr\mathbf{B}$ is a tridiagonal matrix, the systems of (4.8) can be solved by many efficient linear solver such as Thomas' algorithm. Hence, one can see that $O(n)$ operation count is enough to solve the system (4.8). Combining (4.4) with (4.8), one may define the following FSDIRK method for solving the Eqs. (1.1)–(1.3) as

$$\mathbf{U}^{m+1} = \mathbf{U}^m + k\sum_{j=1}^{3} b_j \tilde{\mathbf{K}}_j \tag{4.12}$$

The proposed new scheme can be summarized as follows:

ALGORITHM FSDIRK $(\phi, g_1, g_2, \alpha, \beta, x_L, x_R, h, k, t_0, T)$

1. Remark: The algorithm FSDIRK based on the relations (4.8) and (4.12) is capable for solving the 1D unsteady convection-diffusion Eqs. (1.1)–(1.3). The values of the approximate solution are printed at each time level.
2. Generate the uniform spatial grid nodes vector $\mathbf{x} = [x_L + h, x_L + 2h, \cdots, x_R - 2h, x_R - h]^T$, compute the initial solution, $\mathbf{U}^0 = \phi(\mathbf{x})$ and construct tridiagonal matrices $\mathbf{A}$ and $\mathbf{B}$ defined by (4.7).
3. Let $t_1 := t_0 + k$.
4. If $t_1 > T$, then exit.
5. Compute the vectors $\mathbf{R}_i$ defined by (4.8)
6. Solve the following tridiagonal linear systems

$$(\mathbf{A} - kr\mathbf{B})\tilde{\mathbf{K}}_i = \mathbf{B}\left(\mathbf{U}^0 + k\sum_{j=1}^{i-1} a_{i,j}\tilde{\mathbf{K}}_j\right) + \mathbf{R}_i, \quad i = 1, 2, 3,$$

   where $a_{i,j}$ are defined by (3.3).
7. Calculate

$$\mathbf{U}^1 = \mathbf{U}^0 + k\sum_{j=1}^{3} b_j \tilde{\mathbf{K}}_j,$$

   where b_j are defined by (3.3).
8. Print t_1 and $\mathbf{U}^1$.
9. Set $t_0 := t_1$ and $\mathbf{U}^0 := \mathbf{U}^1$ and then go to step 3.

V. CONVERGENCE ANALYSIS

In this section, we will give a concrete analysis for the convergence of the new scheme (4.12). We begin this section with the convergence analysis of the discretization of the temporal variables. For the solution $u_i(t)(i = 1, 2, \cdots, n-1)$ of (4.3), let us define a $(n-1) \times 1$ vector-valued function $\mathbf{V}(t)$ by $\mathbf{V}(t) := [u_1(t), u_2(t), \cdots, u_{n-1}(t)]^T$. Then, the system (4.3) can be written as

$$\mathbf{V}'(t) = \mathbf{A}^{-1}(\mathbf{B}\mathbf{V}(t) + \mathbf{Q}(t)) \tag{5.1}$$

where $\mathbf{Q}(t)$ is a $(n-1) \times 1$ vector-valued function defined by

$$\begin{aligned}
\mathbf{Q}(t) &= [Q_1(t), 0, \cdots, 0, Q_2(t)]^T, \\
Q_1(t) &= \left(\frac{a}{h^2} + \frac{\alpha}{2h}\right) g_1(t) - \left(\frac{a_2}{h^2} - \frac{a_1}{2h}\right) g_1'(t), \\
Q_2(t) &= \left(\frac{a}{h^2} - \frac{\alpha}{2h}\right) g_2(t) - \left(\frac{a_2}{h^2} + \frac{a_1}{2h}\right) g_2'(t).
\end{aligned} \tag{5.2}$$

Also, we have the following lemma.

Lemma 5.1. *The column vectors $\mathbf{R}_i$ and $\mathbf{Q}(t)$ defined by (4.8) and (5.2), respectively, are related in the following form*

$$\mathbf{R}(\mathcal{A}^T)^i\mathbf{b} = \sum_{j=1}^{4-i} \frac{k^{j-1}}{(j+i)!}\mathbf{Q}^{(j-1)}(t_m), \quad i = 0, 1, 2, 3, \tag{5.3}$$

where $\mathbf{b}$ and $\mathcal{A}$ are defined by (3.3) and $\mathbf{R}$ is a matrix defined by $\mathbf{R} := [\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3]$.

Proof. First we consider the product $[R_{1,1}, R_{2,1}, R_{3,1}](\mathcal{A}^T)^i\mathbf{b}$. Then, from (4.5) and (4.6), the definition of $R_{j,1}(j = 1, 2, 3)$ defined by (4.8) directly gives

$$\begin{aligned}
[R_{1,1}, R_{2,1}, R_{3,1}](\mathcal{A}^T)^i\mathbf{b} &= \left(\frac{a}{h^2} + \frac{\alpha}{2h}\right) \sum_{j=0}^{3-i} k^j g_1^{(j)}(t_m)\mathbf{b}^T\mathcal{A}^{i+j}\mathbf{1} \\
&\quad - \left(\frac{a_2}{h^2} - \frac{a_1}{2h}\right) \sum_{j=1}^{4-i} k^{j-1} g_1^{(j)}(t_m)\mathbf{b}^T\mathcal{A}^{i+j-1}\mathbf{1}.
\end{aligned} \tag{5.4}$$

The definitions of $\mathbf{b}$ and $\mathcal{A}$ defined by (3.3) directly lead to

$$\mathbf{b}^T\mathcal{A}^i\mathbf{1} = \frac{1}{(i+1)!}, \quad i = 0, 1, 2, 3. \tag{5.5}$$

Substituting (5.5) into (5.4) and rearranging the result, we get

$$[R_{1,1}, R_{2,1}, R_{3,1}](\mathcal{A}^T)^i\mathbf{b} = \sum_{j=1}^{4-i} \frac{k^{j-1}}{(i+j)!} \left(\left(\frac{a}{h^2} + \frac{\alpha}{2h}\right) g_1^{(j-1)}(t_m) - \left(\frac{a_2}{h^2} - \frac{a_1}{2h}\right) g_1^{(j)}(t_m)\right). \tag{5.6}$$

Finally, combining (5.6) with the definition of $Q_1(t)$ defined by (5.2), one can get

$$[R_{1,1}, R_{2,1}, R_{3,1}](\mathcal{A}^T)^i \mathbf{b} = \sum_{j=1}^{4-i} \frac{k^{j-1}}{(i+j)!} Q_1^{(j-1)}(t_m). \tag{5.7}$$

Similarly, we have the following relation

$$[R_{1,2}, R_{2,2}, R_{3,2}](\mathcal{A}^T)^i \mathbf{b} = \sum_{j=1}^{4-i} \frac{k^{j-1}}{(i+j)!} Q_2^{(j-1)}(t_m). \tag{5.8}$$

Thus, by (5.7) with (5.8), the Eq. (5.3) is valid. ■

Note that one can simplify the first equation of (4.4) and the Eq. (4.12) in terms of a multiplication of matrices as follows:

$$\mathbf{K} = \mathbf{A}^{-1}\mathbf{B}\boldsymbol{\Theta}^m + k\mathbf{A}^{-1}\mathbf{B}\mathbf{K}\mathcal{A}^T + \mathbf{A}^{-1}\mathbf{R} \tag{5.9}$$

and

$$\mathbf{U}^{m+1} = \mathbf{U}^m + k\mathbf{K}\mathbf{b}, \tag{5.10}$$

where $\mathbf{K}$ and $\boldsymbol{\Theta}^m$ are matrices defined by

$$\mathbf{K} := [\tilde{\mathbf{K}}_1, \tilde{\mathbf{K}}_2, \tilde{\mathbf{K}}_3], \quad \boldsymbol{\Theta}^m := [\mathbf{U}^m, \mathbf{U}^m, \mathbf{U}^m].$$

Based on Lemma 5.1 and the Eq. (5.9), one may have the following lemma.

Lemma 5.2. *For the solution $u_i(t)$ of the system (4.3), we assume that the equalities $u_i^m = u_i(t_m), i = 0, 1, \cdots, n$ are valid. Then, the approximation solution $\mathbf{U}^{m+1}$ of (4.12) has the following local truncation error*

$$\mathbf{U}^{m+1} = \mathbf{V}(t_{m+1}) + O(k^5). \tag{5.11}$$

Proof. From the identity (5.5), one can check that a straightforward calculation gives

$$\boldsymbol{\Theta}^m(\mathcal{A}^T)^i \mathbf{b} = \frac{1}{(i+1)!}\mathbf{U}^m, \quad i = 0, 1, 2, 3, 4. \tag{5.12}$$

Hence, the Eqs. (5.9) and (5.3) give

$$\begin{aligned}\mathbf{B}\mathbf{K}(\mathcal{A}^T)^i \mathbf{b} &= \mathbf{B}\mathbf{A}^{-1}\left(\mathbf{B}\boldsymbol{\Theta}^m(\mathcal{A}^T)^i \mathbf{b} + \mathbf{R}(\mathcal{A}^T)^i \mathbf{b} + k\mathbf{B}\mathbf{K}(\mathcal{A}^T)^{i+1}\mathbf{b}\right)\\ &= \mathbf{B}\mathbf{A}^{-1}\left(\frac{1}{(i+1)!}\mathbf{B}\mathbf{U}^m + \sum_{j=1}^{4-i}\frac{k^{j-1}}{(j+i)!}\mathbf{Q}^{(j-1)}(t_m) + k\mathbf{B}\mathbf{K}(\mathcal{A}^T)^{i+1}\mathbf{b}\right), \quad 0 \le i \le 4.\end{aligned} \tag{5.13}$$

This recurrence relation (5.13) directly gives

$$\mathbf{BK}\mathcal{A}^T\mathbf{b} = \mathbf{BA}^{-1}\left(\frac{1}{2!}\mathbf{BU}^m + \sum_{j=1}^{3}\frac{k^{j-1}}{(j+1)!}\mathbf{Q}^{(j-1)}(t_m) + k\mathbf{BA}^{-1}\left(\frac{1}{3!}\mathbf{BU}^m + \sum_{j=1}^{2}\frac{k^{j-1}}{(j+2)!}\mathbf{Q}^{(j-1)}(t_m)\right.\right.$$
$$\left.\left. + k\mathbf{BA}^{-1}\left(\frac{1}{4!}\mathbf{BU}^m + \frac{1}{4!}\mathbf{Q}(t_m)\right)\right)\right) + O(k^3) \tag{5.14}$$

Rearranging the summation in the right hand side of (5.14) and using (5.1), one can get

$$\mathbf{BK}\mathcal{A}^T\mathbf{b} = \sum_{j=1}^{3}\frac{k^{j-1}}{(j+1)!}\mathbf{BV}^{(j)}(t_m) + O(k^3). \tag{5.15}$$

From the Eqs. (5.3) and (5.12), substituting (5.9) into (5.10) and rearranging the result lead to

$$\mathbf{U}^{m+1} = \mathbf{U}^m + k\mathbf{A}^{-1}\left(\mathbf{BU}^m + \mathbf{Q}(t_m) + \sum_{j=1}^{3}\frac{k^j}{(j+1)!}\mathbf{Q}^{(j)}(t_m)\right) + k^2\mathbf{A}^{-1}\mathbf{BK}\mathcal{A}^T\mathbf{b}. \tag{5.16}$$

Combining (5.16) with (5.15), we get

$$\mathbf{U}^{m+1} = \mathbf{U}^m + k\mathbf{A}^{-1}(\mathbf{BU}^m + \mathbf{Q}(t_m)) + \sum_{j=1}^{3}\frac{k^{j+1}}{(j+1)!}\mathbf{A}^{-1}(\mathbf{Q}^{(j)}(t_m) + \mathbf{BV}^{(j)}(t_m)) + O(k^5) \tag{5.17}$$

Differentiating (5.1) and combining the results with (5.17), the Taylor's expansion of $\mathbf{V}(t_m + k)$ at t_m gives

$$\mathbf{U}^{m+1} = \sum_{j=0}^{4}\frac{\mathbf{V}^{(j)}(t_m)}{j!}k^j + O(k^5) = \mathbf{V}(t_{m+1}) + O(k^5)$$

which complete the proof. ∎

Lemma 5.2 gives only the truncation error for the approximation of the interior values and hence it is required to estimate the truncation error for those of the boundary values, which is performed in the next lemma.

Lemma 5.3. *For the solution $u_i(t)$ of the system (4.3), we assume that the equalities $u_i^m = u_i(t_m), i = 0, 1, \cdots, n$ are valid. Then, the approximation solutions $u_i^{m+1} (i = 0, 1, \cdots, n)$ obtained by (4.12) has the following local truncation error*

$$u_i^{m+1} - u_i(t_{m+1}) = O(k^5), \quad i = 0, n. \tag{5.18}$$

Proof. First, consider the case $i = 0$. Combining the second equation of (4.4) and (4.5) yields

$$u_0^{m+1} = u_0^m + k\mathbf{b}^T[K_{1,0}, K_{2,0}, K_{3,0}]^T = u_0^m + \sum_{j=1}^{4}k^j g_1^{(j)}(t_m)\mathbf{b}^T\mathcal{A}^{j-1}\mathbf{1}. \tag{5.19}$$

By a direct calculation, one see that

$$\mathbf{b}^T \mathcal{A}^{j-1}\mathbf{1} = \frac{1}{j!}, \quad j = 1,2,3,4. \tag{5.20}$$

Hence, by substituting (5.20) into (5.19) and using the Taylor's expansion of $g_1(t)$ at t_m , we get

$$u_0^{m+1} = u_0^m + \sum_{j=1}^{4} \frac{g_1^{(j)}(t_m)}{j!}k^j = \sum_{j=0}^{4} \frac{g_1^{(j)}(t_m)}{j!}k^j = g_1(t_m + k) + O(k^5)$$

which shows that the relation (5.18) is valid in the case $i = 0$. Similarly, one can show that the relation (5.18) is valid in the case $i = n$. ■

Combining (2.10) and (2.14) with the previous two Lemma 5.2 and 5.3, one can prove the following convergence theorem of the proposed method (4.12).

Theorem 5.4. *The approximated solution u_i^{m+1} obtained by using the proposed method (4.12) for (1.1) has the order of convergence $O(h^4, k^4)$.*

VI. STABILITY ANALYSIS

In this section, we will give a concrete analysis of the stability for the new scheme (4.12). For the stability analysis, we consider the problem (1.1) with homogenous boundary conditions. Before proceeding with the discussion of the stability, we see the following results which can be proved by similar procedures of Lemma 4.1.

Corollary 6.1. *The tridiagonal matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *defined in (4.7) are nonsingular.*

From Corollary 6.1 and (4.8), one may check that the homogeneous boundary conditions leads to the values of the internal stages as follows.

$$\begin{aligned}
\tilde{\mathbf{K}}_1 &= \frac{\mathbf{C}}{\mathbf{I} - kr\mathbf{C}}\mathbf{U}^m, \\
\tilde{\mathbf{K}}_2 &= \frac{\mathbf{C}\left(\mathbf{I} + k\left(\frac{1}{2} - 2r\right)\mathbf{C}\right)}{(\mathbf{I} - kr\mathbf{C})^2}\mathbf{U}^m, \\
\tilde{\mathbf{K}}_3 &= \frac{\mathbf{C}(\mathbf{I} + k\mathbf{C}((1-4r)\mathbf{I} + \frac{k}{2}(1 + 2r(7r-4))\mathbf{C}))}{(\mathbf{I} - kr\mathbf{C})^3}\mathbf{U}^m,
\end{aligned} \tag{6.1}$$

where $\boldsymbol{I}$ is the $(n-1) \times (n-1)$ identity matrix and $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$. Hence, combining (4.12) with (6.1) and simplifying the results, one may check that the proposed algorithm (4.12) becomes

$$\mathbf{U}^{m+1} = \mathcal{R}(k\mathbf{C})\mathbf{U}^m, \quad m \geq 0, \tag{6.2}$$

where $\mathcal{R}(z)$ is a rational function defined by

$$\mathcal{R}(z) = \frac{\begin{array}{c}12(1-2r)+z(12(1+r(6r-5))+6z(1-2r(4-9r+6r^2)) \\ +z^2(1+2r(-5+6r(r-1)(2r-3))))\end{array}}{12(1-2r)(1-rz)^3}. \tag{6.3}$$

Let $\mathbf{C} := Q\Lambda Q^{-1}$ *be the eigenvalue decomposition of* $\mathbf{C}$ *and* $\Lambda = diag\,[\lambda_1, \cdots, \lambda_{n-1}]$ *the diagonal matrix, where* λ_j *are the eigenvalues of* $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$*. Then, by the mathematical induction, one can write the Eq. (6.2) as*

$$\mathbf{U}^{m+1} = Q(\mathcal{R}(\Lambda))^{m+1}Q^{-1}\mathbf{U}^0, \quad m \geq 0, \tag{6.4}$$

which converges only if

$$|\mathcal{R}(z)| < 1, \tag{6.5}$$

where $z = k\lambda$ *and* λ *is arbitrary eigenvalue of* $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$*. To show the validity of the inequality (6.5), we recall the following lemma proved by Tian and Yu [14].*

Lemma 6.2. *All eigenvalue of the matrix* $\mathbf{A}^{-1}\mathbf{B}$ *are real and less than zero.*
Hence, by Lemma 6.2, we show that the relation (6.5) is valid in the following lemma.

Lemma 6.3. *For an eigenvalue* λ *of* $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$ *and* $z = k\lambda$ *, the relation (6.5) is valid.*

Proof. Let $\theta_1 = 12(1-2r)+z(12(1+r(6r-5))+6z(1-2r(4-9r+6r^2))+z^2(1+2r(-5+6r(r-1)(2r-3))))$ and $\theta_2 = 12(1-2r)(1-rz)^3$. Then, some manipulation give

$$\begin{aligned}\theta_2^2-\theta_1^2 &= -288z+(576s^2+864s^3)z^2-(24s+480s^2+1296s^3+1008s^4)z^3 \\ &\quad +(12s+204s^2+720s^3+1044s^4+576s^5)z^4+Asz^5-Bz^6, \quad s=2r-1, \\ A &= 6-6s-126s^2-324s^3-360s^4-162s^5, \\ B &= 1+7s+19s^2+21s^3-6s^4-42s^5-45s^6-18s^7. \end{aligned} \tag{6.6}$$

By the definition of r defined by (4.4), one can get the relation $3s^3 = 3s+1$. Hence, the quantities A and B in (6.6) can be simplified as follows.

$$A = -6\left(95+86\sqrt{3}\cos\frac{\pi}{18}+60\cos\frac{\pi}{8}\right) < 0, \quad B = -\frac{167}{3}-44\sqrt{3}\cos\frac{\pi}{18}-\frac{116}{3}\cos\frac{\pi}{9} < 0.$$

Thus, for any negative eigenvalue $\lambda < 0$, each term in the right hand side of the equation $\theta_2^2-\theta_1^2$ in (6.6) is positive and hence one may have $\theta_2^2-\theta_1^2 > 0$ for all $\lambda < 0$, which shows the relation (6.4) is valid. ∎

Lemma 6.3 gives the following theorem.

Theorem 6.4. *The proposed method (4.12) is unconditionally stable.*

TABLE I. Temporal errors and convergence rates for solving Example 7.1 with a fixed spatial step size $h = 0.001$ at $t = 1$.

k	$Err_2(k,h)$	C_{order}	$Err_\infty(k,h)$	C_{order}
$\frac{1}{10}$	1.45e-005	–	1.43e-005	–
$\frac{1}{20}$	1.12e-006	3.7	1.15e-006	3.6
$\frac{1}{40}$	7.59e-008	3.9	8.13e-008	3.8
$\frac{1}{80}$	4.92e-009	3.9	5.39e-009	3.9
$\frac{1}{160}$	3.13e-010	4	3.47e-010	4
$\frac{1}{320}$	2.00e-011	4	2.21e-011	4

VII. NUMERICAL EXPERIMENTS

In this section, we present the numerical results of the proposed method on several problems. We performed our computations in Matlab R2011b software on a Intel(R) Core(TM) i7–2630QM CPU 2.00 GHz with 8.00 GB of memory. To show the accuracy of the proposed method, the global discrete norms of L^2- and $L^\infty-$ errors defined by

$$Err_2(\tau,h) = \sqrt{h\sum_{i=0}^{N+1}(u(T,x_i)-u_i)^2}, \quad Err_\infty(\tau,h) = \max_{x_i}|u(T,x_i)-u_i|, \tag{7.1}$$

respectively, are measured and the rate of convergence C_{order} is calculated by

$$C_{\text{order}} = \log\left(\frac{Err_p(\tau,2h)}{Err_p(\tau,h)}\right), \quad \text{or} \quad C_{\text{order}} = \log\left(\frac{Err_p(2\tau,h)}{Err_p(\tau,h)}\right), \tag{7.2}$$

where $p = 2, \infty$. Here $u(T,x_i)$ and u_i denote the exact solution and the numerical approximation to the model problem at $x = x_i, t = T$, respectively. To show the efficiency of the proposed method, we compare the present method with two existing methods, namely the CNM and the cubic CSCM [17]. For each method, the CPU time (in second) needed to get numerical results with a similar error bound is measured. Because the CSCM requires a matrix solver for a quadratic matrix polynomial consisting of tridiagonal matrices, we adopt the complex tridiagonal matrix solver to get the numerical results for the CSCM as efficiently as possible.

Example 7.1. Consider the following convection-diffusion model

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = 0.01\frac{\partial^2 u}{\partial x^2}(t,x) - \frac{\partial u}{\partial x}(t,x), \quad (t,x) \in (0,\infty)\times[0,2] \\ u(0,x) = \sin(x), \quad x \in [0,2] \end{cases}$$

whose analytic solution is given by $u(t,x) = \exp(-0.01t)\sin(x-t)$(see e.g., [17]).

The boundary conditions of nonhomogeneous type are obtained from the exact solution and the problem is solved on the time interval [1]. Table I shows the temporal discrete norms of L^2- and L^∞ errors and the convergence rate, whenever we fix the spatial step size as $h = 0.001$ and choose different temporal step sizes $k = \frac{1}{n}, n = 10, 20, \cdots, 320$. Whereas, Table II reports the spatial discrete norms of L^2- and $L^\infty-$ errors and the convergence rate with the fixed temporal step size $k = 0.001$ and different spatial step sizes $h = \frac{1}{n}, n = 10, 20, 40, 80, 160$. The numerical

TABLE II. Spatial errors and convergence rates for solving Example 7.1 with a fixed temporal step size k =0.001 at $t=1$.

h	$Err_2(k,h)$	C_{order}	$Err_\infty(k,h)$	C_{order}
$\frac{1}{10}$	1.95e-005	–	2.95e-005	–
$\frac{1}{20}$	1.97e-006	3.3	2.99e-006	3.3
$\frac{1}{40}$	1.55e-007	3.7	2.33e-007	3.7
$\frac{1}{80}$	1.04e-008	3.9	1.57e-008	3.9
$\frac{1}{160}$	6.64e-010	4	9.99e-010	4

TABLE III. Comparisons among CNM, CSCM([17]), and the proposed method for solving Example 7.1.

Method	CNM	CSCM	Proposed method
(k,h)	$(2^{-11},2^{-11})$	$(2^{-11},2^{-6})$	$(2^{-6},2^{-6})$
$Err_2(k,h)$	5.95e-8	3.40e-8	3.48e-8
$Err_\infty(k,h)$	5.79e-8	3.28e-8	4.40e-8
cpu	0.84	0.303	0.0589
(k,h)	$(2^{-14},2^{-14})$	$(2^{-11},2^{-11})$	$(2^{-7},2^{-8})$
$Err_2(k,h)$	9.02e-10	6.53e-10	6.74e-10
$Err_\infty(k,h)$	8.90e-10	7.59e-10	6.46e-10
cpu	48.7	2.1	0.0513

results listed in both Tables I and II achieve the expected fourth-order accuracy in both temporal and spatial variables.

Table III shows the CPU times needed to get a similar error bound for the three methods, CNM, CSCM, and the proposed method, in which the step sizes (k, h) are chosen so that similar error bounds are obtained. One can see that the CSCM requires more smaller CPU time than the CNM to get similar discrete norms of L^∞- and L^2-error bounds, whereas the proposed method is about 40-fold faster than the CSCM in this example.

Example 7.2. Consider the following convection-diffusion equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \beta \frac{\partial^2 u}{\partial x^2}(t,x) - \alpha \frac{\partial u}{\partial x}(t,x), & (t,x) \in (0,2) \times [0,2] \\ u(0,x) = \exp(-\frac{(x-\alpha)^2}{4\beta}), & x \in [0,2] \end{cases}$$

whose analytic solution is given by $u(t,x) = \frac{1}{\sqrt{1+t}}\exp(-\frac{(x-(1+t)\alpha)^2}{4\beta(1+t)})$(see e.g., [22]).

TABLE IV. Temporal errors and convergence rates for solving Example 7.2 with a fixed spatial step size h =0.001 and parameters $\alpha = 0.25, \beta = 0.01$ at time $t=2$.

k	$Err_2(k,h)$	C_{order}	$Err_\infty(k,h)$	C_{order}
$\frac{1}{8}$	1.78e-004	–	2.73e-004	–
$\frac{1}{16}$	1.38e-005	3.7	2.19e-005	3.6
$\frac{1}{32}$	9.55e-007	3.9	1.54e-006	3.8
$\frac{1}{64}$	6.26e-008	3.9	1.01e-007	3.9
$\frac{1}{128}$	4.03e-009	4	6.56e-009	4
$\frac{1}{256}$	2.81e-010	3.8	4.66e-010	3.8

TABLE V. Temporal errors and convergence rates for solving Example 7.2 with a fixed spatial step size h =0.001 and parameters $\alpha = 0.25, \beta = 0.001$ at time $t = 2$.

k	$Err_2(k,h)$	C_{order}	$Err_\infty(k,h)$	C_{order}
$\frac{1}{8}$	1.02e-002	–	2.98e-002	–
$\frac{1}{16}$	1.33e-003	2.9	3.84e-003	3
$\frac{1}{32}$	1.12e-004	3.6	3.22e-004	3.6
$\frac{1}{64}$	7.79e-006	3.8	2.24e-005	3.8
$\frac{1}{128}$	5.12e-007	3.9	1.49e-006	3.9
$\frac{1}{256}$	4.15e-008	3.6	1.29e-007	3.5

TABLE VI. Temporal errors and convergence rates for solving Example 7.2 with a fixed spatial step size h =0.0001 and parameters $\alpha = 0.25, \beta = 0.0001$ at time $t = 2$.

k	$Err_2(k,h)$	C_{order}	$Err_\infty(k,h)$	C_{order}
$\frac{1}{8}$	6.17e-002	–	2.78e-001	–
$\frac{1}{16}$	3.05e-002	1	1.43e-001	0.96
$\frac{1}{32}$	8.22e-003	1.9	4.04e-002	1.8
$\frac{1}{64}$	1.06e-003	3	5.28e-003	2.9
$\frac{1}{128}$	8.21e-005	3.7	4.17e-004	3.7
$\frac{1}{256}$	5.43e-006	3.9	2.77e-005	3.9

This problem has a Gaussian pulse whose sharpness is growing up as the ratio $\frac{\alpha}{\beta}$ is increasing and the problem is solved on the time interval [0,2] with the fixed parameter $\alpha = 0.25$. In three Tables IV–VI, we calculate the convergence rates for the temporal variables with different parameters $\beta = 0.01, 0.001, 0.0001$ and different temporal step sizes $k = \frac{1}{2^n}, n = 3, 4, \cdots, 8$. From the numerical results, one can see that a smaller spatial step size h is required to get the fourth-order accuracy when the problem has a sharp Gaussian pulse.

To show a numerical efficiency, the problem is solved with the fixed parameter $\beta = 0.0001$ at time $t = 2$ and three methods CNM, CSCM, and the proposed method. Table VII shows their global discrete norms of L^2- and $L^\infty-$ errors and the required CPU time to get numerical solutions of Example 7.2. One can see that the proposed method is most efficient among three methods.

Example 7.3. Consider the following convection-diffusion problem which has steep boundary layers near the right boundary

TABLE VII. Comparisons among CNM, CSCM[17], and the proposed method for solving Example 7.2 with $\alpha = 0.25, \beta = 0.0001$.

Method	CNM	CSCM	Proposed method
(k,h)	$(2^{-12}, 2^{-12})$	$(2^{-10}, 2^{-10})$	$(2^{-10}, 2^{-10})$
$Err_2(k,h)$	5.74e-5	6.76e-6	7.36e-6
$Err_\infty(k,h)$	2.78e-4	3.80e-5	4.14e-5
cpu	7.4	2.28	0.829
(k,h)	$(2^{-14}, 2^{-14})$	$(2^{-12}, 2^{-12})$	$(2^{-10}, 2^{-10})$
$Err_2(k,h)$	3.59e-6	2.64e-8	3.15e-8
$Err_\infty(k,h)$	1.74e-5	1.49e-7	1.77e-7
cpu	176	17.9	11.8

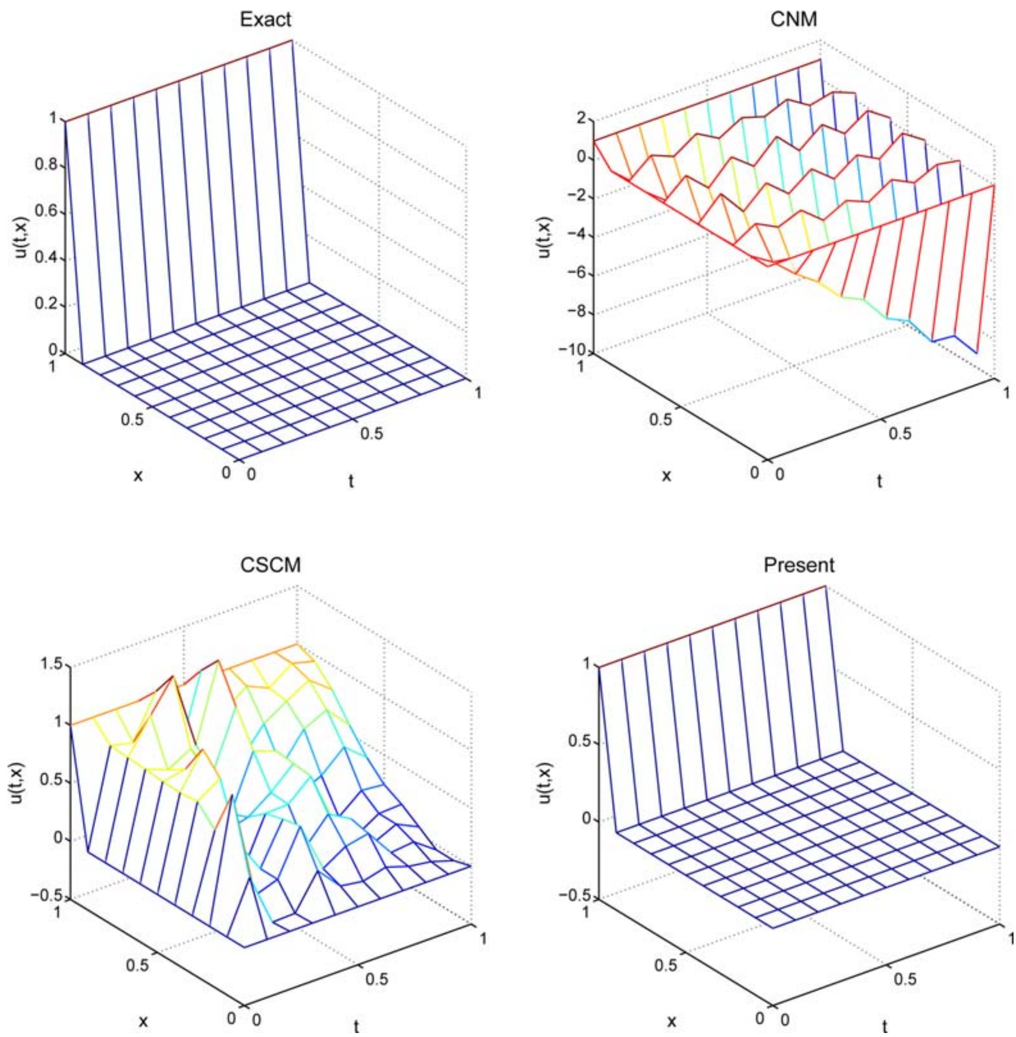

FIG. 1. Comparisons of numerical solutions using CNM, CSCM [17], and the proposed scheme for solving Example 7.3 with $k = h = 0.1$ and $\mu = 10^{-4}$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \mu \frac{\partial^2 u}{\partial x^2}(t,x) - \frac{\partial u}{\partial x}(t,x), & (t,x) \in (0,1) \times [0,1] \\ u(0,x) = 0, \quad x \in (0,1) \\ u(t,0) = 0, \quad u(t,1) = 1, t > 0. \end{cases}$$

The analytical solution of this problem is given by (see e.g., [14])

$$u(t,x) = \frac{\exp(\frac{x}{\mu}) - 1}{\exp(\frac{1}{\mu}) - 1} + \sum_{k=0}^{\infty} \frac{(-1)^k k\pi}{(k\pi)^2 + \frac{1}{4\mu^2}} \exp\left(\frac{(x-1)}{2\mu} \sin(k\pi x) \exp\left(-((k\pi)^2\mu + \frac{1}{4\mu}\right) t\right).$$

In this example, we let $\mu = 1.0e-4$. Solution surfaces are shown in Figs. 1 and 2 with different space grid sizes $h = 0.1$ and $h = 0.01$. Note that the results computed by the present scheme are

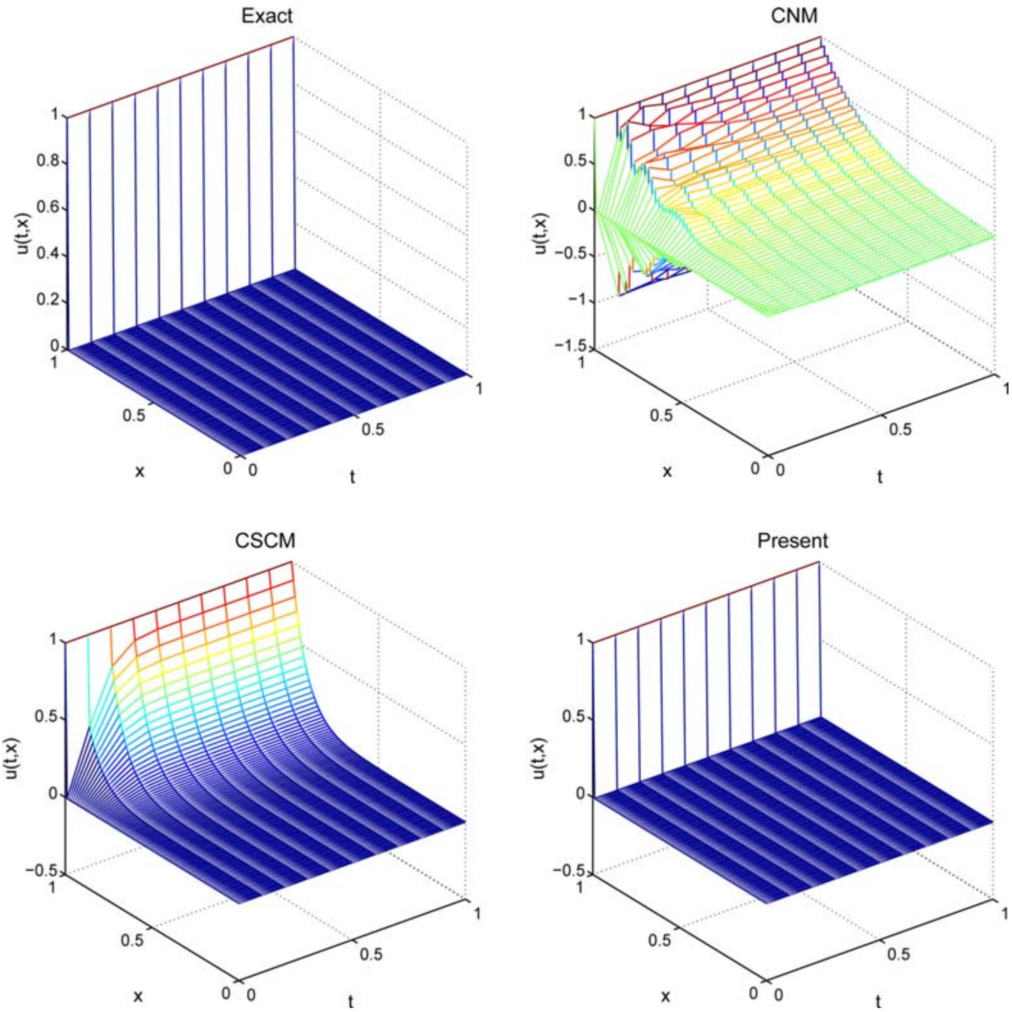

FIG. 2. Comparisons of numerical solutions using CNM, CSCM [17], and the proposed scheme for solving Example 7.3 with $k = 0.1, h = 0.01$ and $\mu = 10^{-4}$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

in full agreement with exact solutions with without nonphysical oscillation. However, both CNS and CSCM [17] produce the results with nonphysical oscillations.

Example 7.4. Consider the following nonlinear convection-diffusion equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \mu \frac{\partial^2 u}{\partial x^2}(t,x) - u(t,x)\frac{\partial u}{\partial x}(t,x), & (t,x) \in (0,1) \times [0,1] \\ u(0,x) = \sin(\pi x), & x \in (0,1) \\ u(t,0) = 0, \quad u(t,1) = 0, t > 0. \end{cases}$$

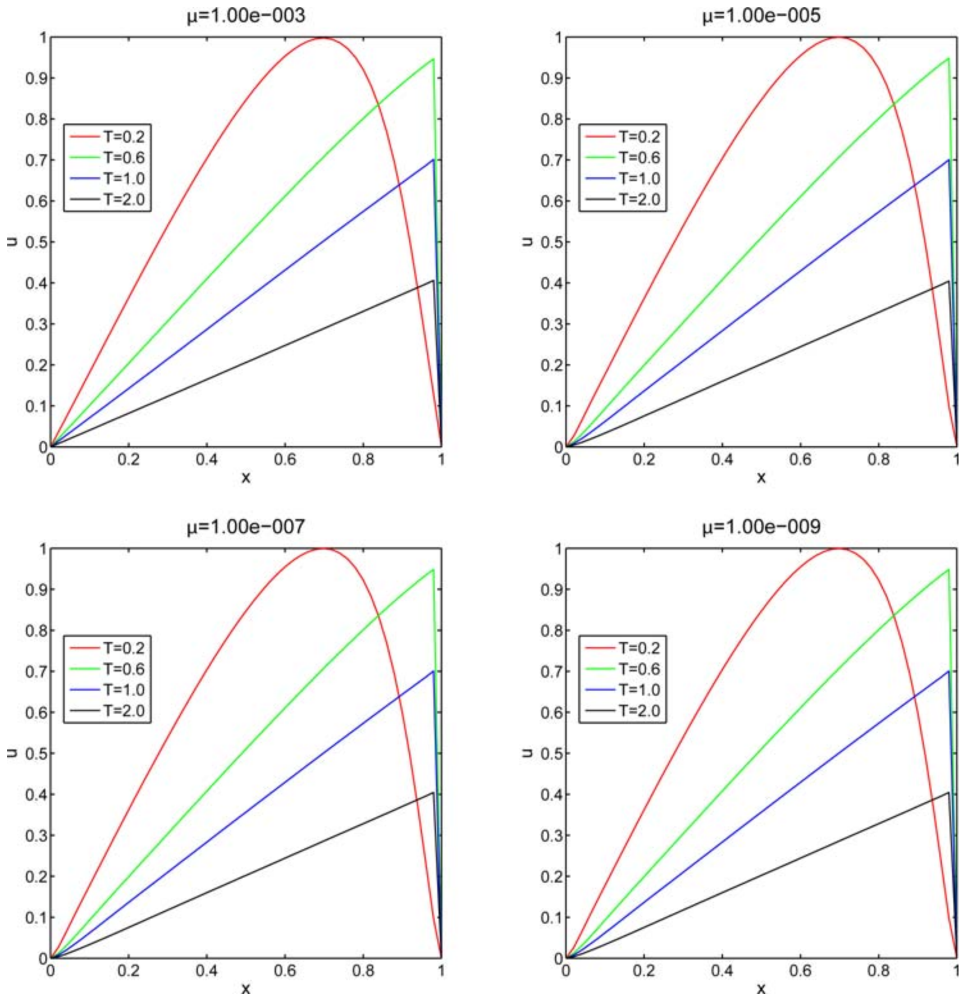

FIG. 3. Comparisons of numerical solutions using the proposed scheme for solving Example 7.4 with fixed $h = 0.02$ and $k = 0.05$ when varying the parameter μ. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

which is the well-known Burgers equation and its analytic solution given by(see e.g., [14])

$$u(t,x) = 4\pi\mu \frac{\sum_{n=1}^{\infty} \exp(-n^2\pi^2\mu t) I_n(\frac{1}{2\pi\mu}) n \sin n\pi x}{I_0(\frac{1}{2\pi\mu}) + 2\sum_{n=1}^{\infty} \exp(-n^2\pi^2 t\mu) I_n(\frac{1}{2\pi\mu}) \cos n\pi x}.$$

where $I_n(x)(n = 0, 1, 2, \cdots)$ are the modified Bessel function of first kind.

To solve the problem, instead of the original problem, we consider the following linearized Burgers' equation with constant coefficient

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \mu \frac{\partial^2 u}{\partial x^2}(t,x) - u^n(x) \frac{\partial u}{\partial x}(t,x), 0 < x < 1, t_n \le t \le t_{n+1} \\ u(t_n, x) = u^n(x), \quad x \in (0,1) \\ u(t,0) = 0, \quad u(t,1) = 0, t_n \le t \le t_{n+1}. \end{cases}$$

TABLE VIII. Two errors $\mathrm{Err}_2(\mathrm{k,h}), \mathrm{Err}_\infty(\mathrm{k,h})$ between the exact solution and the numerical solution for example 7.4 with fixed $h = 0.02, k = 0.05$, and $\mu = 10^{-2}$.

T	Tian and Yu[14]		Proposed method	
	$Err_2(k,h)$	$Err_\infty(k,h)$	$Err_2(k,h)$	$\mathrm{Err}_\infty(\mathrm{k,h})$
0.2	0.01461	0.03937	0.01476	0.03909
0.6	0.01393	0.02204	0.01238	0.03269
1.0	0.01060	0.02394	0.01061	0.02209
2.0	0.00507	0.00833	0.00512	0.00833

This is the convection-diffusion type equation, which can be solved by the proposed scheme. Computations, using the proposed scheme, are implemented for varying the parameter μ on uniform grids of sizes $h = 0.02$ with time step size $k = 0.05$. Figure 3 shows the numerical solutions at different times for $\mu = 10^{-3}, 10^{-5}, 10^{-7}$ and $\mu = 10^{-9}$. It is observed the proposed scheme demonstrates the correct physical behavior for different μ. In Table VIII, we estimate two errors $Err_2(k,h)$ and $Err_\infty(k,h)$ for the numerical solution obtained by the present scheme and the method of Tian and Yu [14] with the fixed step sizes $h = 0.02, k = 0.05$ and the parameter $\mu = 10^{-2}$ at different times $T = 0.2, 0.6, 1.0, 2.0$. One may see that two methods have a similar performance.

VIII. CONCLUSIONS

The proposed FSDIRK method is designed to solve unsteady 1D convection-diffusion Eq. (1.1). We combine a three point fourth-order compact finite difference scheme for the spatial variable and a three-stage SDIRK method for the temporal variable. Also, a formulation evaluating the boundary values assigned to the internal stages for the SDIRK method is derived, which does not give a phenomenon of a reduction of the convergence order. The proposed scheme requires only a linear solver for tridiagonal matrix systems, which can solve (1.1) more efficiently than using many existing methods. It is proved that the method is unconditionally stable and has the fourth-order accuracy in both space and time variables. Throughout several numerical experiments, we give the numerical evidence for the convergence analysis in Theorem 5.4 and show the superiority of the new method over existing methods: the CNM and a cubic C^1 spline collation method [17].

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